AD-A115 393 STANFORD UNIV CA DEPT OF STATISTICS F/G 12/1
ON ASYMMETRIC STOCHASTIC BANG-BANG CONTROL.(U)
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UNCLASSIFIED TR-316 NL

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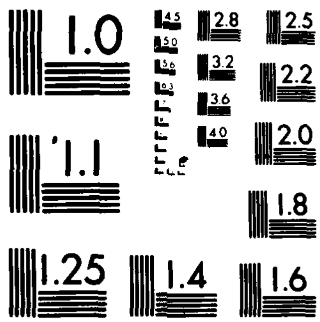

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ON ASYMMETRIC STOCHASTIC BANG-BANG CONTROL

By

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TECHNICAL REPORT NO. 316

April 6, 1982

Prepared Under Contract
N00014-76-C-0475 (NR-042-267)
For the Office of Naval Research

Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

On Asymmetric Stochastic Bang-Bang Control

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I. Introduction

Three stochastic bang-bang control problems are considered, the predicted miss, the linear regulator, and a simple complete observation model ([1],[2],[3]), which have been solved for symmetric constant bound on the control function $u(t)$, $0 \leq t \leq T$. Here asymmetric, finite bounds on $u(t)$ are considered.

To state these problems we use the notation and definitions of ([1],[2],[3]).

II. Predicted Miss

Let $A(t)$, $B(t)$, $C(t)$ be, respectively, dxd, rxd, and dxd matrix-valued continuous functions on $[0,T]$ with

$$\langle \underline{\alpha}, C(t)C'(t)\underline{\alpha} \rangle \geq \beta \langle \underline{\alpha}, \underline{\alpha} \rangle > 0$$

for all $t \in [0,T]$ and $\underline{\alpha} \in R^d$, where $\langle \underline{a}, \underline{b} \rangle = \sum_{i=1}^{d} a_i b_i$.

Denote the system equation by

$$dX(t) = A(t)X(t)dt + B(t)u(t)dt + C(t)dW(t)$$

$$X(0) = G \in R^d \tag{1}$$

and W(t) is standard d-dimensional Wiener process. Transpose A is indicated by A'.

Assumption: The admissible control set $\mathcal{C}$ consists of the set of processes

$u(t) \equiv u(t,w)$ such that $-\infty < a < f(t) \leq u(t,w) \leq g(t) < b < \infty$

where $f(t) < g(t)$ are bounded continuous non-random given functions on $[0,T]$, and $f(0) + g(0) = 0$.

The Girsanov Theorem may be used to solve (1) in law (i.e. there is a weak solution. The boundedness of u(t,w) insures that for any given $u \in \mathcal{C}$, there is a unique solution to (1) by ([4]), Theorem 1).

Given a fixed vector $\underline{\gamma}$, the cost corresponding to $u \in \mathcal{C}$ is

$$J(u) = E_u(\ell(<\gamma,X(T)>))$$

where for each u, there is a probability space $(\Omega,\mathcal{F},P_u)$ with $\Omega = C^d[0,T]$, $X(t,w) : \Omega \to R^d$ is the coordinate map $X(t,w) = w(t)$. Then for $u \in \mathcal{C}$, P_u on $(\Omega,\mathcal{F})$, where $\mathcal{F} = \sigma(X(s),\ 0 \leq s \leq T)$ is such that $P_u[X(0)=G] = 1$ and the process W(t,u) defined by

$$W(t,u) = \int_0^t C^{-1}(s)dX(s) - \int_0^t C^{-1}(s)[A(s)X(s)+B(s)U(s)]ds \text{ is}$$

d-dimensional Wiener process. Hence

$$J(u) \equiv \int_\Omega \ell(<\gamma, X(T)>)\ dP_u = E_u\ell(<\gamma, X(T)>).$$

where

$$\ell : R \to R^+ \text{ has these properties:}$$

(i) ℓ is even : $\ell(x) = \ell(-x)$.

(ii) ℓ is continuously differentiable for $x > 0$

and $\ell'(x) \geq 0$ all $x > 0$

(iii) $\ell(x) = O(\exp \delta|x|)$, some $\delta > 0$.

The object is to find optimal $u_0(t)$ such that

$$J(u_0) = \min_{u \in \mathcal{Q}} J(u).$$

Continuing the exposition in [2], if $X(t) = x$, and $u(\xi) = 0$, $t \leq \xi \leq T$,

then

$$E[\langle \gamma, X(T) \rangle | X(t) = x] = \langle \gamma, \Phi'(t,T)x \rangle$$

where Φ is the solution operator for (1) when $u = 0$.

Define

$$s(t) = \Phi'(t,T)\gamma.$$

Then $s(t)$ satisfies

$$\frac{ds(t)}{dt} = -A'(t)s(t), \qquad s(T) = \gamma .$$

and

$$E_u[\langle \gamma, X(T) \rangle | X(t)] = \langle s(t), X(t) \rangle \equiv m(t).$$

Then $m(t)$ satisfies

$$dm(t) = \langle B'(t)s(t), u(t) \rangle dt$$

$$+ \langle C'(t)s(t), dW(t,u) \rangle$$

or, equivalently,

$$dm(t) = \langle b(t), u(t) \rangle dt + dv(t)$$

$$m(0) = \langle s(0), G \rangle \qquad (2).$$

The cost is expressed as

$$J(u) = E_u[\ell(m(T))]$$

<u>Theorem 1</u> Under conditions above in II, the optimal control $u_0(t)$ is expressible by components, $1 \leq i \leq d$, with $u_0 = (u_{01}, \ldots u_{0d})$, as

$$u_{0i}(t) = \frac{f_i(t) + g_i(t)}{2}$$

$$- \frac{(g_i(t)-f_i(t))}{2} \operatorname{sign}\left([m(t) - \frac{1}{2}\sum_{\ell=1}^{d}\int_t^T b_\ell(s)(f_\ell(s)+g_\ell(s))ds]b_i(t)\right) \qquad (3)$$

Proof. First assume $f_i(t) + g_i(t) = 0$, all $1 \leq i \leq d$ and set

$$h_i(t) = \frac{g_i(t) - f_i(t)}{2}$$

In this case, note $0 < h_i(t) \leq |a|+|b|$, $i \leq i \leq d$ and by [4], since $|u_i(t)| \leq h_i(t)$, then (2) has a unique solution with $u = u_0$.

Set

$dm(t) = \langle b,u\rangle dt + dv$ then it follows by the same argument as in [2] that

$$u_{0i}(t) = -(h_i(t) \operatorname{sign}\left(m(t)b_i(t)\right). \qquad (4)$$

is optimal for the symmetric control case. In general, by the argument of ([2], pp. 207-208), one may invoke symmetry by utilizing a

switching curve $k(t)$ such that if

$$\bar{m}(t) \equiv m(t) - k(t), \text{ then it would follow that}$$

$$d\bar{m}(t) = \langle b, u - \frac{f+g}{2} \rangle dt + dv(t). \tag{5}$$

To accomplish (5) it clearly suffices to set

$$\frac{dk(t)}{dt} = \frac{1}{2} \sum_{\ell=1}^{d} b_\ell(t)(f_\ell(t)+g_\ell(t))$$

and $$k(T) = 0,$$

which allows $$J(\bar{m}(T)) = J(m(T)). \tag{6}$$

Hence (4) - (6) suffice for the proof of (3).

III. Linear Regulator.

A one-dimensional linear regulator problem, following [3] is defined as follows: The one-dimensional process $X(t,w)$ is given by

$$dX(t,w) = (aX(t,w) + u(t,w))dt + dW_1(t,w)$$

with

$$X(0,w) = X_0(w)$$

and observation equation

$$dY(t,w) = cX(t,w)dt + dW_2(t,w) \tag{7}$$

and

$$Y(0,w) = 0$$

for

$$0 \leq t < T$$

where

$a > 0$, $c > 0$ are constants, W_1, W_2 are independent

one-dimensional Wiener processes. Let $(\Omega,\mathcal{F},P_u,)$ be $\Omega = C[0,T]$, $\mathcal{F} = \sigma(X(s),\ 0 \le s \le T)$, and P correspond to W_1,W_2.

Let the performance index, as a function of a given control u be

$$J(u) = \int_0^T E(X^2(s))ds \equiv \int_0^T X^2(s,u)dP \tag{8}$$

and the set of admissible controls $\mathcal{A}$ is given by

$$\mathcal{A} = \{u|\ |u(t,w)| \le g(t), \quad 0 \le t \le T, \tag{9}$$

with continuous $g(t) > 0$, $0 \le t \le T$.

A control $U_0 \in \mathcal{A}$ is optimal if

$$J(u_0) \le J(u) \text{ all } u \in \mathcal{A}.$$

This problem may be recast as a complete observation control problem, following ([3], eq. (3.19) - (3.21)). The new state variables are $R(t,w)$ and satisfy

$$dR(t,w) = (aR(t,w) + u(t,w))dt + c\ p(t)\ dW_3(t,w)$$

$$R(0,w) = E\ X_0(w)$$

and
$$J(u) = \int_0^T E(R^2(s))ds \tag{10}$$

where W_3 is a Wiener process, and the function $p(t)$ satisfies a Riccati equation

$$\frac{dp}{dt} = 2ap(t) + 1 - c^2p^2(t) \qquad 0 \le t \le T$$

$$p(0) = E(X_0^2) - (EX_0)^2 = \mathrm{Var}X_0. \tag{11}$$

The admissible control set $\mathcal{A}$ is unchanged and it is noted that (11) does not depend on $\mathcal{A}$ by ([3], eq. (2.15) - (2.20)).

Also $J(u) = \int_0^T E(R^2(s))ds$

<u>Theorem 2</u> The equation, for any fixed $u \in \mathcal{A}$,

$$dV(t,w) = (aV(t,w) + u(t,w))dt + cp(t)dW_3(t,w)$$

$$V(0,w) = EX_0(w) \tag{12}$$

has a unique solution.

<u>Proof</u>. This follows from the boundedness of p, u, f, g in [0,T] by ([4], Theorem 1).

<u>Theorem 3</u> The optimal $u \in \mathcal{A}$ for the system (10), (11) is expressible as

$$u_0(t,w) = -g(t)\text{sign } X(t,w). \tag{13}$$

<u>Proof</u>. This follows since $g(t) > 0$ factors out of both sides of ([3], eq. (2.26)), and Theorem 2.

IV Complete Observation

Consider a one-dimensional complete observation control problem with state X(t,w), control u(t,w) and Wiener process W(t,w) defined by

$$dX(t,w) = u(t,w)dt + dW(t,w) \tag{14}$$

and

$$X(0,w) = x \ .$$

The $(\Omega \mathcal{F} P)$ are as in III.

With admissible control set

$$\mathcal{a} = \{u \mid |u(t,w)| \leq g(t), \quad 0 \leq t \leq T\}$$

with continuous $g(t) > 0$, $0 \leq t \leq T$, as in (9).

The performance index for given $u \in \mathcal{a}$

is

$$J(u) = \int_0^T E|X(t,w)|^{\ell} dt \tag{15}$$

for ℓ a fixed positive integer. The object is to find $u_0 \in \mathcal{a}$ so that

$$J(u_0) \leq J(u), \ u \in \mathcal{a}.$$

For $\ell = 1, 2$, this problem was solved in [1].

<u>Theorem 4</u> Under (9), (14), (15), for all $\ell \geq 1$, the solution is

$$u_0^o(t,w) = -g(t)\text{sign } X(t,w). \tag{16}$$

<u>Proof</u>. The equation, with $X(0,w) = x$

$$X(t,w) = -g(t)\text{sign } X(t,w)dt + dW(t,w)$$

has a unique solution ([4], Theorem 1) by boundedness of g.

The reasoning of ([1], pp. 93, 96, eq. (2.15)) implies that the same optimal u_0 holds for all $\ell \geq 1$ in (15). The argument is as in Theorem 3.

REFERENCES

1. Balakrishnan, A. V. (1980) On stochastic bang-bang control. Applied Mathematics and Optimization 6, 91-96.

2. Davis, M. H. A. and Clark, J. M. C. (1979) On "predicted miss" stochastic control problems. Stochastics Vol. 2, 197-209.

3. Ruzicka, J. (1975) On a class of stochastic bang-bang control problems. Lecture Notes in Economics and Mathematical Systems: Control Theory, Numerical Methods and Computer System Modelling no. 107, 250-261.

4. Zvonkin, A. K. (1974). A drift annihilating phase space map for a diffusion. Matematiceskii Sbornik, Tom 93 (135), No. 1, 129-149 (in Russian).

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 316	2. GOVT ACCESSION NO. AD-A115 393	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON ASYMMETRIC STOCHASTIC BANG-BANG CONTROL		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) HOWARD J. WEINER		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0475
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS Office Of Naval Research Statistics & Probability Program Code 411SP Arlington, VA 22217		12. REPORT DATE April 6, 1982
		13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Asymmetric, Bang-Bang, Optimal Stochastic Control, Stochastic Differential Equations.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Three stochastic bang-bang control problems, the predicted miss, the linear regulator, and a complete observation model are shown to have formally similar solutions under asymmetric boundedness condition on the control $u(t)$, $0 < t < T$.

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S/N 0102-LF-014-6601

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